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VIBRATION AND STABILITY OF A CANTILEVER COLUMN SUBJECT TO A FOLLOWER FORCE PASSING THROUGH A FIXED POINT

L. Tomski, J. Przybylski, M. Gołębiowska-Rozanow and J. Szmidla

Institute of Mechanics and Machine Design Foundations, Technical University of Częstochowa, Dabrowskiego 73, 42-200 Częstochowa, Poland

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The paper describes the stability and natural vibration of a column loaded by a force passing through a fixed point. It is proved that the system is conservative and its loss of stability occurs via divergence. On the other hand a change in modes occurring along the fundamental eigencurve, which modifies its slope from positive to negative with the increasing load, is found theoretically and confirmed by the modal analysis. For that reason the system is proposed to be called the divergence–pseudo-flutter system. Experimental results concerning the natural vibration frequency agreed well with the numerical ones.

1. INTRODUCTION

The stability behaviour of elastic systems such as columns and frames determines their classification into one of two types: flutter systems (FT), for which the natural frequency curves coalesce at the point of instability as in Figure 1(a) [1], or divergence type systems (DT), of the divergence instability [2] occurring when the natural frequency takes values equal to zero [Figure 1(b)].

Cantilever columns loaded by a follower force can change the critical instability mechanism from flutter to divergence or vice versa, depending on certain parameters that are included in the boundary conditions. Such values are, for example, a follower parameter [3], the rigidity of a linear or rotational spring [4, 5], as well as the nonlinear (cubic) member of the translational spring rigidity [6]. The joint effect of the follower parameter and the linear spring rigidity upon the type of instability is presented in reference [7]. All systems described in the above mentioned papers are either the divergence–flutter type (D-F), or flutter–divergence (F-D), or divergence–flutter–divergence type systems (D-F-D). That depends on their type of instability which occurs for a case when some parameters included in the boundary conditions take values equal to zero. Leipholz [8] determined conditions under which structures may be classified as conservative of the first and second kind. Conservative structures are always of the divergence type; and damping can be neglected when investigating their stability.

The literature concerning theoretical investigations of the flutter and divergence type systems is vast [9–12]. However there is a lack of work concerning elastic systems for which the variation of the natural frequency against the external load is given in Figure 1(c). This variation has been obtained theoretically and supported by an experiment for cantilever columns loaded by a certain type of general loading, and presented in reference [13]. These systems cannot be treated in the sense described in reference [2] as the flutter or divergence type systems.

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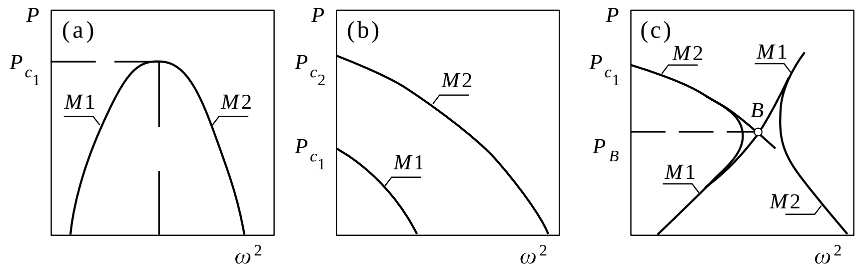

Figure 1. Eigenvalue curves of: (a) flutter type structure; (b) divergence type structure; and (c) divergence–pseudo-flutter type structure.

The new systems' feature is, that for the external load $P \to P_c$, where P_c is the divergence critical load, the eigenvalue curve has a negative slope whereas for $P \in <0$, P_c, that slope can be negative, zero, or positive. Moreover, for these systems along their eigenvalue curves a change in eigenmodes appears, as it is for the system of the eigencurves sketched in Figure 1(c), where $M1$ and $M2$ describe the first and second mode, respectively. For that reason the new systems can be called the divergence–pseudo-flutter systems [14, 15]. An experimental verification of theoretical results obtained for a divergence–pseudo-flutter column has been presented in references [13, 16] and for an identically loaded planar frame.

It should be noticed that in references [18, 19] experimental variation of the natural frequency curves of a type as in Figure 1(c), have been demonstrated for columns of an unknown loading scheme.

An idea of the elastic system presented in this work was inspired by two different columns described in references [20, 21]. For the first one—Beck's column [20] which is presented in Figure 2(a), its compressive force is tangent to the deflected end of the column (the angle of force P inclination to the vertical is $\beta_a = W'(l, t)$, where $'$ denotes differentiation with respect to x). Boundary conditions for this column are presented in

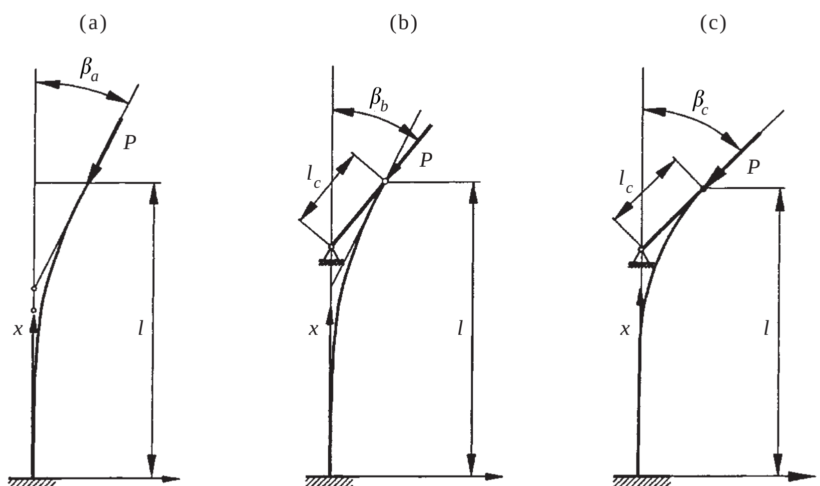

Figure 2. Schemes of: (a) Beck's column: $\beta_a = W'(l, t) \neq W(l. t)/l_c$, $W''(l. t) = W'''(l, t) = 0$; (b) column loaded by a force passing through a fixed point: $\beta_b = W(l, t)/l_c \neq W'(l, t)$, $EJW'''(l, t) + P[W'(l, t) - W(, t)/l_c] = 0$, $W''(l, t) = 0$; and (c) column of present investigation: $\beta_c = W'(l, t) = W(l, t)/l_c$.

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the figure caption. Such a load appears, for example, for structures exposed to streaming media [22], as well as for a clamped-free [23, 24] or a free-free column [25] loaded by a rocket trust. Experimental investigations of such problems were published in references [23, 24] where it was stated that a cantilever column can lose its stability by flutter, i.e. oscillations with increasing amplitudes. An influence of a variety of parameters such as elastic spring supports, concentrated masses, transverse shear deformation as well as rotary inertia on the stability of a cantilever column subjected to a tangential load has been discussed by Kounadis [26].

The second system which was described in reference [21] and caused the present investigation is a column which is compressed by a force passing through a fixed point independently from the deflection of the column [Figure 2(b)]. The distance of that point from the free end of the column is taken to be >0 for the case as in Figure 2(b) or <0 when the fixed point is placed above the point of force application [27]. The angle of the compressive force is $\beta_b = W(l, t)/l_c$. A variation of the natural vibration curves for the system depends on the value of l_c and can be as in Figure 1(b) [28] or as in Figure 1(c) [28, 29]. Constructional variants of a column loaded by a force passing through a fixed point were given in references [30, 31] where experimental models were developed to simulate Beck's column. The theoretical analysis given in reference [29] led to a conclusion that a conservative system associated with Beck's column may be found to study its dynamic behaviour.

Taking into account the above considerations a question was asked concerning the possibility of constructing such a real system which has features characteristic of columns from Figures 2(a) and (b). This system is presented in Figure 2(c) together with the boundary condition for angle β_c. Two different variants of the system are schematically drawn in Figures 3(a) and (b). Experiments presented in this study concern the column from Figure 2(b); the second boundary condition for $x = l$ is given in the third part of the work [equation (7)].

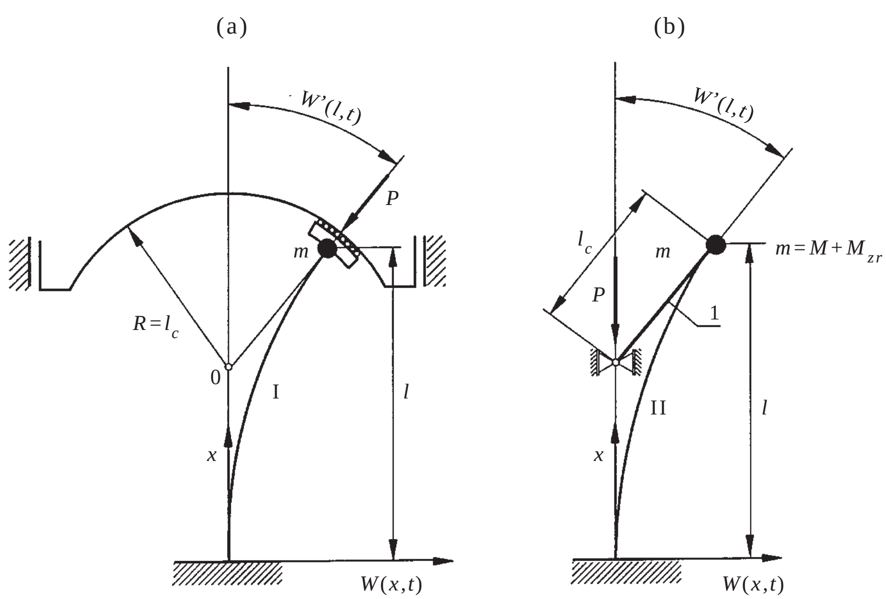

Figure 3. Constructional variants of columns of the same boundary conditions and loading, but different way of force application: (a) load passing through a fixed point and; (b) column loaded by a moment.

2. STATEMENT OF THE PROBLEM

In this work the stability and natural vibration of a cantilever column subjected to a follower force passing through a fixed point has been considered. This load can be realised in two constructional variants shown in Figures 3(a, b). Column I [Figure 3(a)] is loaded by a force P tangent to the deflection at the point of the force application ($x = l$), and passing through a fixed point O. A stiff element 1 of the length l_c of column II—Figure 3(b), is carrying a vertical force P. Hence for these columns the boundary conditions are

$$W(l, t) = l_c W'(l, t) \tag{1}$$

$$W(0, t) = W'(0, t) = 0. \tag{2}$$

At the free end of both columns a concentrated mass m is mounted. For column II mass m is composed of the concentrated mass M and the reduced mass M_{zr} of the element 1. The point of reduction is the point of mass M fixing.

The present investigation has been directed toward the following objectives:

- in the field of theory
  - —to analyse the total elastic energy of columns from Figure 3 to verify whether each system is conservative or nonconservative;
  - —to evaluate the divergence critical force and its maximum as a function of the length l_c.
  - —to reveal on the basis of eigenvalue curves that the columns are the divergence–pseudo-flutter systems (D-PF);
- in the field of experiment and numerical calculations
  - —to present an experimental and construction of the column from Figure 3(b);
  - —to perform experimental investigations of the natural vibration frequencies as a function of an external load for two columns of different geometrical and physical properties;
  - —to analyse the change in vibration modes along the eigenvalue curves.

3. CONSERVATIVENESS OF THE SYSTEM, BOUNDARY CONDITIONS

For the foregoing system the Hamilton principle has the form [32]

$$\delta \int_{t_1}^{t_2} \left[\sum_{i=1}^{2} (T_i - V_i) + L \right] \mathrm{d}t = 0 \tag{3}$$

where

$$T_1 = \tfrac{1}{2}\rho A \int_0^l [\dot{W}(x, t)]^2 \, \mathrm{d}x \tag{3a}$$

is the kinetic energy of the column,

$$T_2 = \tfrac{1}{2} m [\dot{W}(l, t)]^2 \tag{3b}$$

is the kinetic energy of the concentrated mass,

$$V_1 = \frac{EJ}{2} \int_0^l [W''(x, t)]^2 \, \mathrm{d}x \tag{3c}$$

is the energy of the elastic deformation,

$$V_2 = -\frac{P}{2}\int_0^l [W'(x, t)]^2 \, dx \tag{3d}$$

is the potential energy of the vertical component of the force P,

$$L_A = -PW(l, t)W'(l, t), \qquad L_B = -PW(l, t)W'(l, t) \tag{3e, f}$$

are the work of the horizontal component of the force P for column from Figure 3(a), or the work of the bending moment for column from Figure 3(b), respectively, ρA is the mass per unit length, EJ is the flexural rigidity.

Taking into account the geometric condition (1), one obtains

$$L_A = L_B = L = -P\frac{W^2(x, t)}{l_c}. \tag{3g}$$

Due to the form of equation (3g), an analogous derivation as in reference [33] makes it possible to establish the potential of the horizontal force P component or the bending moment as follows

$$V_3 = \frac{P}{2}\frac{W^2(x, t)}{l_c} \tag{4}$$

with

$$\delta V_3 = -\delta L, \qquad \delta L = -PW\delta W. \tag{5a, b}$$

Existence of the potential (4) indicates that the foregoing system is conservative.

After introducing equations (3a–g) into (3), integrating by parts, performing the variational operations one obtains the Bernoulli–Euler equation in the form

$$W^{IV}(x, t) + \lambda W''(x, t) + \frac{\rho A}{EJ}\ddot{W}(x, t) = 0 \tag{6}$$

as well as the boundary condition

$$W'''(l, t) - \frac{1}{l_c}W''(l, t) - \frac{m}{EJ}\ddot{W}(l, t) = 0. \tag{7}$$

Here $\lambda = P/EJ$ is the external load parameter.

Differential operators $(\cdot)^{IV}$ and $(\cdot)^{II}$ from equation (6) are self-adjoint due to the boundary conditions (1–2) and (7); hence the first derivative of the energy-functional is equal to zero [2], which is another proof that the system is a conservative one.

The solution of equation (6) for a column performing small vibrations is

$$W(x, t) = y(x)\, e^{i\omega t} \tag{8}$$

where ω is the natural vibration frequency.

4. DIVERGENCE CRITICAL LOAD

The divergence critical load parameter $\lambda_c = P_c/EJ$ may be obtained from the condition of setting to zero the total energy of the system, variation of which is expressed by equation

(3), for vanishing value of the natural frequency ($\omega = 0$) occurring in equation (8). This leads to

$$\lambda_c = \frac{\int_0^l [y''(x)]^2 \, \mathrm{d}x}{\int_0^l y'^2(x) \, \mathrm{d}x - y(l)y'(l)}. \tag{9}$$

Solving equation (6) for $\omega = 0$ with the consideration of boundary conditions (1, 2) and (7) leads to the transcendental equation for critical load parameter:

$$G(l_c, \sqrt{\lambda_c}) = l\sqrt{\lambda_c} \cos(l\sqrt{\lambda_c}) + [l_c\lambda_c(l - l_c) - 1] \sin l\sqrt{\lambda_c} = 0 \tag{10}$$

The derivative of the above function with respect to l_c is as follows

$$\begin{aligned}\frac{\mathrm{d}l\sqrt{\lambda_c}}{\mathrm{d}l_c} &= -\frac{\partial G/\partial l_c}{\partial G/\partial l\sqrt{\lambda_c}} \\ &= \frac{\lambda_c(l - 2l_c)\sin l\sqrt{\lambda_c}}{\lambda_c l_c l(l_c - l)\cos l\sqrt{\lambda_c} + \lambda_c(2l_c^2 + l^2 - 2l_c l)\sin l\sqrt{\lambda_c}}.\end{aligned} \tag{11}$$

This derivative becomes zero for $l_c = l/2$, which results in appearing at this point a maximum of function (10).

5. EIGENMODES $\Omega(\lambda)$

Consider the functional [2]

$$F[y] = \int_0^l (yy^{IV} + \lambda yy'' - \Omega y^2) \, \mathrm{d}x = 0 \tag{12}$$

here $y \equiv y(x)$, $\Omega = \omega^2/EJ$.

The variation of this functional with utilization of equation (8) leads to the following

$$\delta F[y] = \int_0^l (yy''\delta\lambda - y^2\delta\Omega) \, \mathrm{d}x + \int_0^l y(\delta y^{IV} + \lambda\delta y'' - \Omega\delta y) \, \mathrm{d}x. \tag{13}$$

In the second part of equation (13) the variations of both the second and fourth derivatives appear; a form which has been obtained by integration by parts. After that and with the application of the boundary conditions one obtains

$$\frac{\mathrm{d}\Omega}{\mathrm{d}\lambda} = \frac{-\int_0^l y'^2(x) \, \mathrm{d}x + y'(l)y(l)}{\int_0^l y^2(x) \, \mathrm{d}x + \dfrac{m}{\rho A} y^2(l)}. \tag{14}$$

In the above expression the differential notation replaces the variational notation.

Substituting (9) into (14) for $\lambda \to \lambda_c$ one may write

$$\frac{\mathrm{d}\Omega}{\mathrm{d}\lambda} = -\frac{\int_0^l [y''(x)]^2 \, \mathrm{d}x}{\lambda_c\left(\int_0^l y^2(x) \, \mathrm{d}x + \dfrac{m}{\rho A} y^2(l)\right)} < 0. \tag{15}$$

It it obvious from (15) that for $\lambda \to \lambda_c$ the slope of the eigenvalue curve $\lambda = \lambda(\Omega)$ is always negative.

The deflected shapes of columns for the extreme values of the length of the stiff element l_c are sketched in Figures 4(a) and (c). In a certain range of $l_c \in (l_{c1}, l_{c2})$ the column must

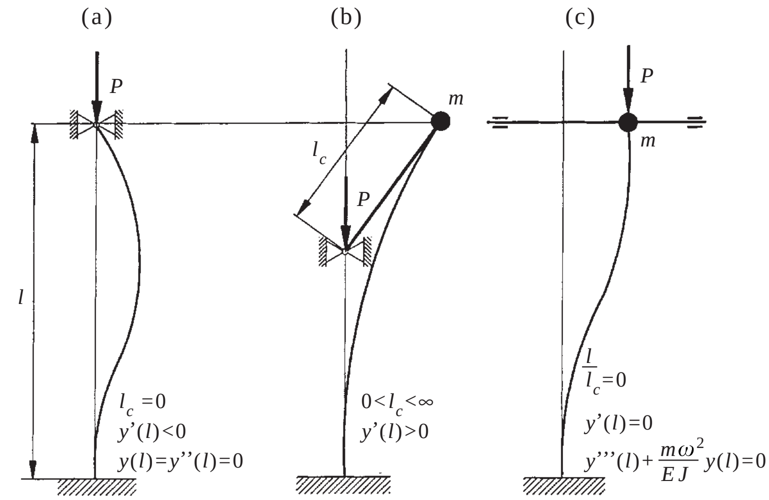

Figure 4. Columns of different length of the loading link: (a) $l_c = 0$; (b) $0 < l_c < \infty$; and (c) $1/l_c = 0$.

have the deflected shape as in Figure 4(b), for which the first derivative $y'(x)$ is monotonic for the external load parameter λ growing from zero.

On the basis of the theorem of the average value of an integral of the monotonic function, the numerator of expression (14) is greater than zero and can be presented in the following form

$$N = -\int_0^l y'^2(x)\,\mathrm{d}x + y'(l)y(l) = y'(l)y(\xi) > 0 \tag{16}$$

where $\xi \in \langle 0, l \rangle$.

For the increasing value of λ, the variation of $y'(x)$ changes from monotonic to non-negative, and hence the numerator of expression (14) takes the form

$$N = [-y'(\xi) + y'(l)]y(l) \tag{17}$$

and can be less, or greater, or equal to zero.

When performing analogous considerations for a further increase of λ, for which the second vibration modes appears, it can be proved that the numerator N is negative. This is in accordance with the above analysis of equation (14) for $\lambda \to \lambda_c$, which reveals that only the numerator N in (14) can be less than zero in the considered range of λ, which results in the negative slope of the natural frequency curve.

Since for small values of λ the numerator $N > 0$, whereas for $\lambda \to \lambda_c$ the numerator $N < 0$, so it must exist such a boundary value of $\lambda = \lambda_B = P_B/EJ$, for which N and also (14) is equal to zero [P_B is marked in Figure 1(c)].

6. EXPERIMENT

The schematics of the experimental set-up are sketched in Figure 5(a). The presented stand may be used for the natural vibration tests of both columns and planar frames under different boundary conditions by using two mounted loading heads 1(1) and 1(2). Head 1(1) can be horizontally shifted in two perpendicular directions along guides 2(1). Similarly, head 1(2) can be horizontally slid along guides 2(2), as well as transversally along

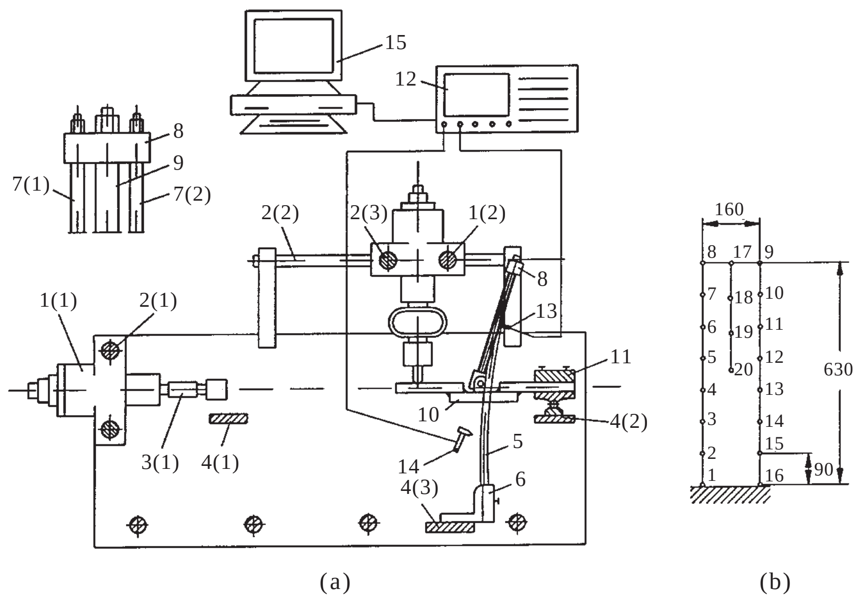

Figure 5. Experimental set-up: (a) for measuring frequency; and (b) for modal analysis.

guides 2(3). Load to the column is applied by means of built-in screw systems and measured by extensometers 3(1) and 3(2). Required boundary conditions are set in supports fixed to plates 4(1), 4(2) and 4(3). The first results of tests and theoretical investigations of the dynamics of systems in the aspect of different boundary conditions have been published in references [13, 15, 34–36]. The column (5) which is compound of two identical rods 7(1) and 7(2) of the circular cross section, is clamped at one end by the holder (6). At the other end, both rods are connected by means of the rigid element (8) of mass M. In this way both displacements and deflection angles of the rods as well as the loading link (9), which is placed in the middle of the rods, are to be equal. The bending rigidity of the link (9) is much greater than this of the column bars. The link is joined by a needle bearing to the beam (10), which is mounted in the holder (11) pinned to the plate 4(2). A force created in the head 1(2) is transferred to the column (5) by means of the beam (10).

Vibration tests were performed with the use of a two-channel vibration analyser (12) of 2035 type and the accelerometer (13) of 4381 type made by Brüel & Kjaer. The system was activated by the manual impactor (14).

In order to study the experimental vibration modes, the virtual discrete model of the system composed of 19 elements joined by 20 nodes adequate to the real system was created also [Figure 4(b)]. An accelerometer (4381 type) was mounted at the node number 11 to collect system responses for the excitement of nodes 1–20. The measured band of both the excitement and responses was in the range 0–400 Hz within the measuring resolution of 1 Hz. Five natural frequencies were identified within the obtained response band. The modal analysis was done with the help of the PC MODAL program (Vibration Engineering Consultants, U.S.A.) run on a personal computer (15) coupled to the analyser (12).

7. SOLUTION OF THE PROBLEM

The equations of motion and boundary conditions for the column of the bending rigidity EJ are stated in the first part of the work (1, 2, 6, 7).

To describe correctly the column from the experiment its mathematical model is also composed of two rods. The problem is geometrically linear if both the flexural and axial rigidities of one rod are equal to those of the second rod, respectively (i.e. $E_1J_1 = E_2J_2$, $E_1A_1 = E_2A_2$).

To fulfil the boundary conditions (1) the loading link (9) from Figure 5(a) has its bending rigidity much greater then the rigidity of the sum of two column rods (7).

The equations of motion for the column composed of two identical rods are

$$E_iJ_i \frac{\partial^4 W_i(x,t)}{\partial x^4} + S\frac{\partial^2 W_i(x,t)}{\partial x^2} + \rho_i A_i \frac{\partial^2 W_i(x,t)}{\partial t^2} = 0 \qquad (i = 1, 2) \tag{18}$$

where $E_1J_1 = E_2J_2 = \frac{1}{2}EJ$, $\rho_1 A_1 = \rho_2 A_2 = \frac{1}{2}\rho A$, $S = P/2$.

For the small vibration one can apply

$$W_i(x,t) = y_i(x)\,e^{i\omega t}. \tag{19}$$

After separation of the time and space variables one derives

$$EJy_i^{IV}(x) + Py_i''(x) - \rho A\omega^2 y_i(x) = 0. \tag{20}$$

The boundary conditions are as follows

$$y_1(0) = y_2(0) = y_1'(0) = y_2'(0) = 0 \tag{21a–d}$$

$$y_1'(l) = y_2'(l), \qquad y_1(l) = y_2(l) \tag{22a, b}$$

$$y_1(l) = l_c y_1'(l) \tag{23}$$

$$y_1'''(l) + y_2'''(l) - \frac{l}{l_c}(y_1''(l) + y_2''(l)) + \frac{m\omega^2}{EJ} = 0. \tag{24}$$

The general solution of equations (20) is

$$y_i(x) = C_{1i}\cosh(\alpha x) + C_{2i}\sinh(\alpha x) + C_{3i}\cos(\beta x) + C_{4i}\sin(\beta x) \tag{25}$$

where C_{ji} is the integration constant ($j = 1, 2, 3, 4$), and $\alpha^2 = -0{\cdot}5\lambda + (0{\cdot}25\lambda^2 + \rho A\Omega)^{1/2}$, $\beta^2 = 0{\cdot}5\lambda + (0{\cdot}25\lambda^2 + \rho A\Omega)^{1/2}$.

After introducing equations (25) into the boundary conditions (21–24) one gets a system of eight homogenous equations. Equating the determinant of the matrix coefficients of this system to zero leading to the transcendental equation for the eigenvalues of the column.

8. NUMERICAL AND EXPERIMENTAL RESULTS

The solution of the boundary problem of the cantilever column composed of two rods and loaded by a force passing through a fixed point leads to conclusions concerning the natural frequencies and modes.

Three characteristic vibration modes can be distinguished here as a result of the investigation (see Figure 6).

Mn—modes ($n = 1, 2, \ldots$), which are characterized by the property that the nth mode has $(n-1)$ nodes along the length of the column. These modes are identical with the modes of a single column of the bending rigidity EJ and mass per unit length ρA;

Mn^e—symmetric mode. Its feature is that the vibration node appears at the point $x = l$ and that $y(l) = y^{I}(l) = 0$. The number of nodes for the n-mode is $(n-1)$. Such modes are characteristic for columns formed by two rods. It is easy to foresee that the values of the natural frequencies cannot depend on the value of the concentrated mass m as well as the length of the stiff element l_c when columns take symmetric modes. The relevant frequencies

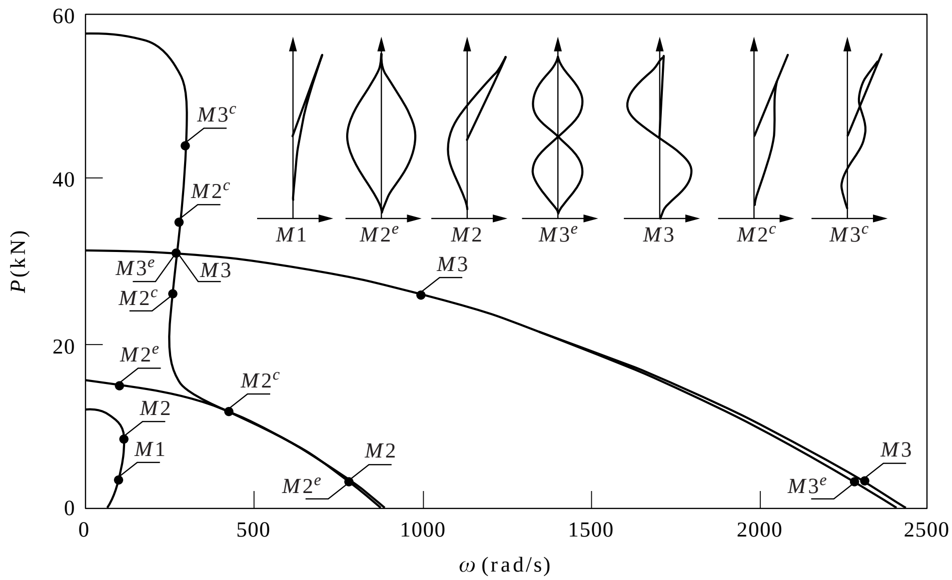

Figure 6. Eigenvalue curves for column A (Table 1) together with the first seven modes investigated at points marked along curves.

are equal to the frequencies for the single column with both ends clamped, and depend upon the bending rigidity EJ and mass per unit length ρA. Such modes were obtained for a column with pinned ends and presented in reference [36];

Mn^c—modes characterized by $(n-2)$ nodes for the nth mode. These modes are identical with the modes for a single column of the bending rigidity EJ and mass per unit length ρA.

The natural frequency curves sketched in Figure 6 obtained numerically for column A from Table 1, are presented together with the first seven natural vibration modes which occur along these curves. It is noticeable that the change in the modes does not appear along curves Mn^e but it is present along other curves. The existence of mode shapes $M1$, $M2^e$, $M2$, $M3^e$, $M3$ has been experimentally confirmed during the modal analysis of this system loaded by a force $P = 0$ and $P = 7840$ N.

Experimental tests of the natural frequency as a function of the external load have been performed for two columns (A and B) from Table 1. The numerical and experimental results are in a good agreement (Figure 7). The maximum relative difference between calculated and tested fourth vibration frequency for column B was equal to 10%. Results concerning the third vibration frequency for column A are marked with bars because there

TABLE 1

Geometrical and physical parameters of column

Column	Rod's diameter (m)	$E_1 = E_2$ (MPa)	$\rho_1 = \rho_2$ (kg/m^3)	m (kg)
A	0·012	$7{\cdot}5 \times 10^4$	2790	0·59
B	0·014	$7{\cdot}5 \times 10^4$	2790	0·60
	$l = 0{\cdot}63$ m;	$l_c = 0{\cdot}31$ m		

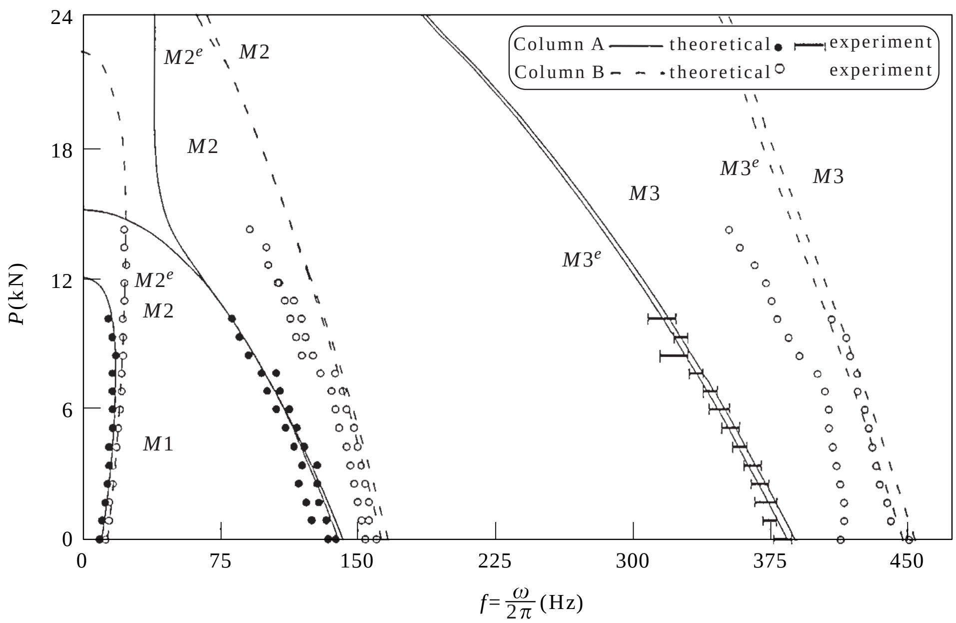

Figure 7. Comparison of experimental and numerical results of the natural frequencies.

were no one peak values on the analyser display for these frequencies. The length of each bar is equal to the width of the measured frequency band.

In Figure 8 the dimensionless critical load parameter λ_c^* versus the relation of $c = l_c/l$ is given ($\lambda^* = \lambda l^2$). The values λ_{ca}^* and λ_{cc}^* are the critical parameters for columns from Figures 4(a) and (c), respectively. The solid line represents the change in the critical

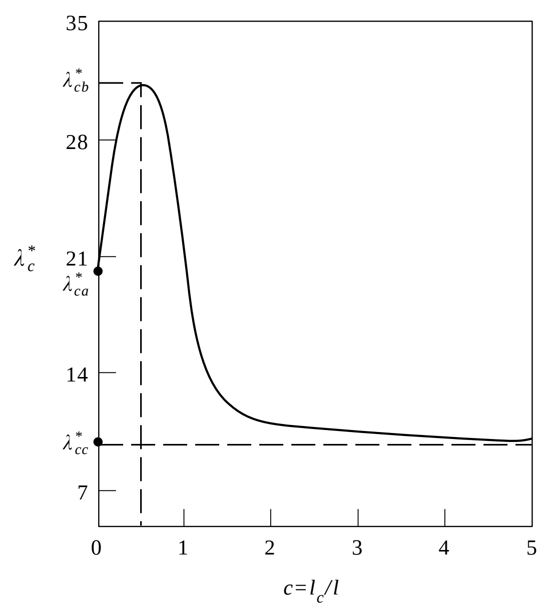

Figure 8. Effect of the length of loading link l_c on the critical load parameter.

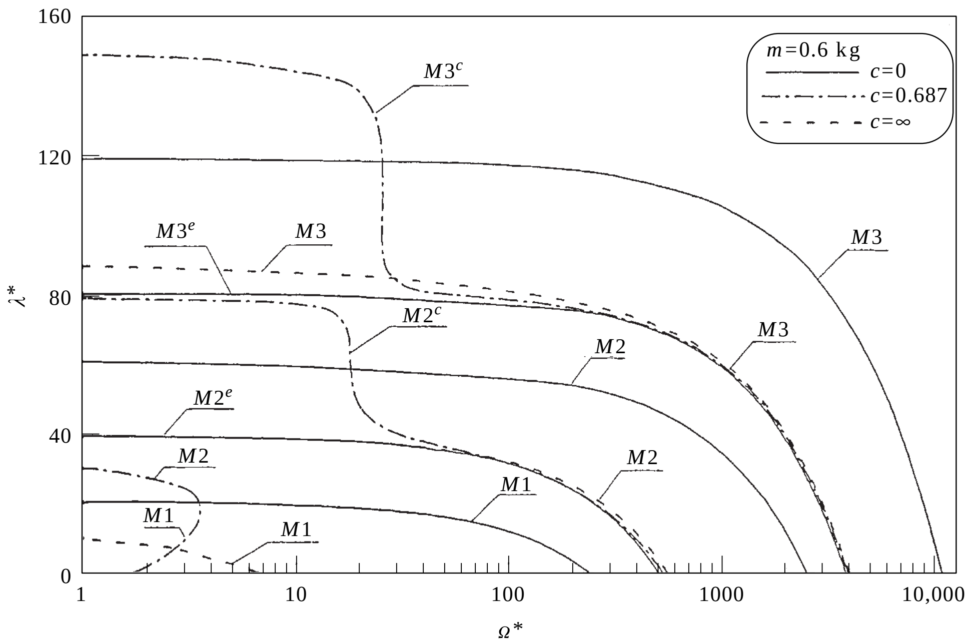

Figure 9. Eigenvalue curves for columns of different length of loading link l_c ($c = l_c/l$).

parameter for the considered column [Figure 4(b)]. Accordingly to both the formula (11) and Figure 8, the maximum critical force exists for $l_c = l/2$.

In Figure 9 the dimensionless natural frequencies Ω^* are plotted against parameter λ^* for columns from Figures 4(a, b, c) of different dimensionless length c of the loading member ($\Omega^* = \Omega \rho A l^4$, $c = l_c/l$). All curves related to the symmetric modes Mn^e are not

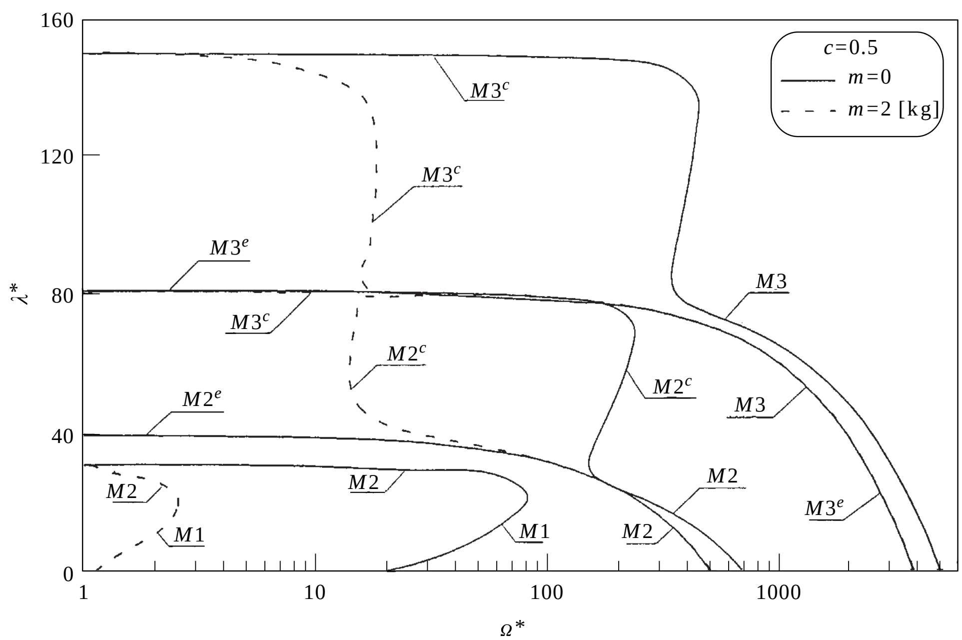

Figure 10. Eigenvalue curves for columns of different concentrated mass m.

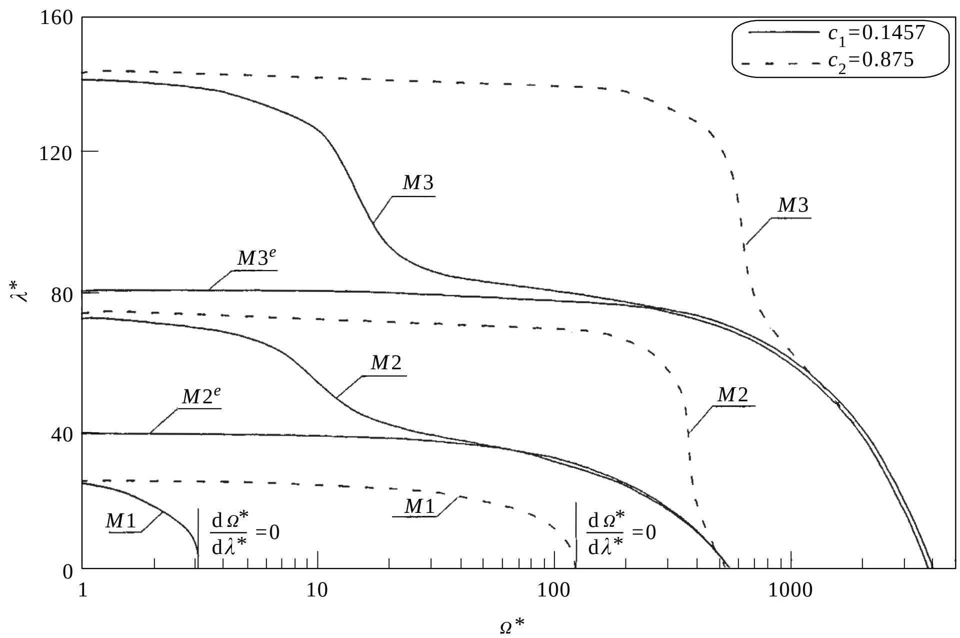

Figure 11. Eigenvalue curves for boundary values of parameter c.

affected by the length l_c so they overlap each other. The change in the modes appears along curves obtained for $c = 0{\cdot}687$ only, because only this system belongs to the class of divergence–pseudo-flutter systems.

Similar phenomenon of common natural frequencies independently from the concentrated mass m magnitude occurs also for the symmetric modes Mn^e and is shown in Figure 10. Other variations of the frequency curves depend on the mass m, however the divergence critical loads are unchanged for any mass.

Numerical analysis confirmed the existence of such border values $c_1 = l_{c1}/l$ and $c_2 = l_{c2}/l$ for which the first derivative $d\Omega^*/d\lambda^*$ for the first eigencurve is equal to zero for $\lambda^* = 0$. The natural frequency curves related to these values of c are presented in Figure 11. For $0 \leqslant c \leqslant c_1$ or $c_2 \leqslant c \leqslant \infty$, the columns are the divergence type systems whereas for $c \in (c_1, c_2)$ the divergence–pseudo-flutter type system.

9. CONCLUSIONS

The analysis of the total energy of a column loaded by a force passing through a fixed point proves that the system is conservative.

Two constructional variants of the follower force passing through a fixed point for a column have been found. Such a column can be either of the divergence–pseudo-flutter type for the length of the loading link $l_c \in (l_{c1}, l_{c2})$, or the divergence type if $0 \leqslant l_c \leqslant l_{c1}$ or $l_{c2} \leqslant l_c \leqslant \infty$. The border values of l_c have been established also. The value of the divergence critical force is affected by the length l_c. The maximum load occurs for $l_c = 0{\cdot}5l$.

For the considered column built of two identical rods additional symmetric modes appear which do not exist for a single column of flexural stiffness equal to the sum of the flexural stiffnesses of both rods. Those modes and their frequencies do not depend on the value of the concentrated mass m as well as the length of the loading link l_c.

In the range of performed experiments, test results concerning the natural vibration frequencies confirmed those from the theoretical investigation and computation with a good agreement.

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